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EXTENSION OF AN ELASTIC SPACE WITH A RIGID BAR

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An asymptotic model for deformation of an elastic space with a rigid thin reinforcing bar is constructed. The elastic modulus of the fiber far exceeds the elastic modulus of the matrix. The shape optimization problem for the reinforcing bar is solved on the basis of the uniform strength condition.

Key words: *elastic space, reinforcing bar, asymptotic model, uniform strength condition.*

1. Formulation of the Problem. Let an elastic (with Young's modulus E and Poisson's ratio ν) space contain an absolutely rigid, thin, cylindrical inclusion $Q_\varepsilon = \omega_\varepsilon \times (-l, l)$ extended along the Ox_3 axis. (The case of an elastic inclusion of variable cross section is considered below.) Here ε is a small positive parameter and ω_ε is a circle of radius

$$r_\varepsilon = \varepsilon l. \tag{1.1}$$

It is assumed that the elastic space is extended by a stress $\sigma_{33} = \sigma$ applied at infinity.

This problem was studied by Nikoshkov and Cherepanov [1], who obtained an approximate analytical and numerical solutions. Mirenkova and Sosnina [2] and Kachalovskaya and Ulitko [3] obtained asymptotic solutions for the case of a rigid inclusion of ellipsoidal shape. A similar problem was solved by Phan-Thien [4]. In addition to an ellipsoidal inclusion, Kanaun [5] (see also [6, § 4.3]) considered a cylindrical inclusion and an inclusion in the shape of a tapered spindle. Using the method of [7], which was originally employed to solve the contact problem of beam bending in an elastic half-space, Khait [8] obtained an approximate solution for the problem of an elastic cylindrical inclusion An asymptotic solution of the heat-conduction problem was obtained by Fedoryuk [9] and Mazya et al. [10]. Zorin and Nazarov [11, 12] constructed an asymptotic solution of the elastic problem for the case of a absolutely rigid, toroidal inclusion.

In the present paper, we use the asymptotic method proposed by Argatov and Nazarov [13, 14]. A refined asymptotic model is constructed using a modified joining procedure [15].

2. External Asymptotic Representation. The extension of an elastic space is described by the linear displacement field

$$\boldsymbol{v}^0(\boldsymbol{x}) = (\sigma/E)[-\nu(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2) + x_3\boldsymbol{e}_3]. \tag{2.1}$$

By virtue of symmetry, the displacements of the points belonging to the surface of a rigid bar should be zero. Hence, in the boundary conditions on the lateral surface of the bar Q_ε, the first two components of the vector (2.1) yield the residual $O(\varepsilon)$, and the third component the residual $O(1)$.

The effect of the inclusion on the deformation of the matrix is determined in the main by tangential stresses on its lateral surface; therefore, at a distance from the inclusion, the displacement field of the points of the elastic space differs insignificantly from the field

$$\boldsymbol{v}(p; \boldsymbol{x}) = \boldsymbol{v}^0(\boldsymbol{x}) + \int_{-l}^{l} \boldsymbol{T}^{(3)}(x_1, x_2, x_3 - s)p(s)\,ds. \tag{2.2}$$

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Here $p(s)$ is a function that characterizes the response of the bar; $\boldsymbol{T}^{(3)}(\boldsymbol{x})$ is Kelvin solution of the problem of an elastic space acted upon by a unit concentrated force oriented along the Ox_3 axis, and (see, for example, [16, § 5.7])

$$T_k^{(3)}(\boldsymbol{x}) = \frac{1}{M}\Big(\frac{x_3 x_k}{|\boldsymbol{x}|^3} + (3-4\nu)\,\frac{\delta_{3,k}}{|\boldsymbol{x}|}\Big) \quad (k=1,2,3), \qquad M = \frac{8\pi E(1-\nu)}{1+\nu}. \tag{2.3}$$

According to (2.3), the singular component of the vector (2.2) has the following components:

$$M\int_{-l}^{l} T_k^{(3)}(\boldsymbol{y}, z-s)p(s)\,ds = -y_k I_3^1(p;\boldsymbol{y},z) \qquad (k=1,2); \tag{2.4}$$

$$M\int_{-l}^{l} T_3^{(3)}(\boldsymbol{y}, z-s)p(s)\,ds = 4(1-\nu)I_1^0(p;\boldsymbol{y},z) - |\boldsymbol{y}|^2 I_3^0(p;\boldsymbol{y},z). \tag{2.5}$$

Here

$$I_m^n(p;\boldsymbol{y},z) = \int_{-l}^{l} \frac{(s-z)^n p(s)\,ds}{[(s-z)^2+|\boldsymbol{y}|^2]^{m/2}}.$$

For $|\boldsymbol{y}| = (y_1^2+y_2^2)^{1/2} \to 0$, the behavior of the vector function $\boldsymbol{v}(p;\boldsymbol{x})$ is determined by the following asymptotic formulas, which are written under the assumption of smooth density $p(z)$:

$$I_3^1(p;\boldsymbol{y},z) = O\Big(C_0\sum_{\pm}\frac{1}{l\pm z} + C_1\ln\Big(\frac{\sqrt{l^2-z^2}}{|\boldsymbol{y}|}\Big) + C_2 l\Big); \tag{2.6}$$

$$I_3^0(p;\boldsymbol{y},z) = \frac{2}{|\boldsymbol{y}|^2}\,p(z) + O\Big(C_0\sum_{\pm}\frac{1}{(l\pm z)^2} + C_1\sum_{\pm}\frac{1}{l\pm z} + C_2\ln\Big(\frac{\sqrt{l^2-z^2}}{|\boldsymbol{y}|}\Big) + C_3 l\Big); \tag{2.7}$$

$$I_1^0(p;\boldsymbol{y},z) = p(z)\Big[-2\ln\Big(\frac{|\boldsymbol{y}|}{2l}\Big) + \ln\Big(1-\frac{z^2}{l^2}\Big)\Big] + (\boldsymbol{J}p)(z) + O\Big(C_1|\boldsymbol{y}| + C_0\sum_{\pm}\frac{|\boldsymbol{y}|^2}{(l\pm z)^2}\Big). \tag{2.8}$$

Here

$$C_i = \max_{z\in[-l,l]} p^{(i)}(z) \qquad (i=0,1,2).$$

The integral operator $\boldsymbol{J}$ acts according to the formula

$$(\boldsymbol{J}p)(z) = \int_{-l}^{l}\frac{p(s)-p(z)}{|z-s|}\,ds. \tag{2.9}$$

We note that for a fixed value of $z\in(-l,l)$, only the third component $v_3(p;\boldsymbol{y},z)$ of the vector (2.2) is unbounded for $|\boldsymbol{y}|\to 0$ [see, in particular, relations (2.8) and (2.5)]. The main terms of the asymptotic representations of integrals (2.4) and (2.5), which are defined by formulas (2.6)–(2.8), agree with the results of [4].

3. Plane Boundary Layer. In planes orthogonal to the bar axis, we introduce the extended coordinates

$$\boldsymbol{\eta} = (\eta_1,\eta_2), \qquad \eta_i = \varepsilon^{-1}y_i. \tag{3.1}$$

The approximate representation of the displacement field of the matrix near the inclusion is sought in the form

$$\boldsymbol{V}(\boldsymbol{\eta};z) = V_3(\boldsymbol{\eta};z)\boldsymbol{e}_3. \tag{3.2}$$

We substitute expression (3.2) into the Lamé equation and separate higher-order terms with respect to the parameter ε. Then, in the exterior of a closed circle $\overline{\omega}_1$ of radius l, the function $V_3(\boldsymbol{\eta};z)$ should satisfy the Laplace equation

$$\mu\Delta_\eta V_3(\boldsymbol{\eta};z) = 0, \qquad \boldsymbol{\eta}\in\mathbb{R}^2\setminus\overline{\omega}_1, \tag{3.3}$$

where $\mu = E[2(1+\nu)]^{-1}$ is the shear modulus.

In the case of an absolutely rigid bar, Eq. (3.3) is supplemented by the homogeneous boundary condition

$$V_3(\boldsymbol{\eta}; z) = 0, \qquad \boldsymbol{\eta} \in \partial\omega_1. \tag{3.4}$$

In problem (3.3), (3.4), the dependence on the extended variable $z \in (-l, l)$ is parametric.

According to the method of joined asymptotic expansions (see, for example, [17, 18]), the asymptotic formulas (2.5) and (2.8) define the logarithmic behavior of the function $V_3(\boldsymbol{\eta}; z)$ as $|\boldsymbol{\eta}| \to \infty$.

The function

$$G_\infty(\boldsymbol{\eta}) = -\frac{1}{2\pi} \ln\Big(\frac{|\boldsymbol{\eta}|}{l}\Big) \tag{3.5}$$

satisfies relations (3.3) and (3.4), and

$$\int\limits_{\partial\omega_1} \partial_\nu G_\infty(\boldsymbol{\eta})\, ds_\eta = 1. \tag{3.6}$$

Here $\partial_\nu = \nu_1(\partial/\partial\eta_1) + \nu_2(\partial/\partial\eta_2)$ is the derivative along the inward (with respect to the region ω_1) unit normal $\boldsymbol{\nu} = (\nu_1, \nu_2)$ to the contour $\partial\omega_1$, and ds_η is an element of the arc length.

We set

$$V_3(\boldsymbol{\eta}; z) = \mu^{-1} p(z) G_\infty(\boldsymbol{\eta}). \tag{3.7}$$

Then, the tangential stresses in the elastic space near the lateral surface of the bar Q_ε are calculated by the formulas

$$\sigma_{i3}(\boldsymbol{V}; \boldsymbol{y}, z) = \varepsilon^{-1} p(z) \frac{\partial G_\infty}{\partial\eta_i}(\boldsymbol{\eta}) \qquad (i = 1, 2). \tag{3.8}$$

From formulas (3.8) and (3.6), the resultant tangential stress $\sigma_{n3} = \sigma_{13}n_1 + \sigma_{23}n_2$ normalized to the unit length of the bar on an area with the inward (with respect to Q_ε) unit normal $\boldsymbol{n} = (n_1, n_2, 0)$, is equal to

$$\int\limits_{\partial\omega_\varepsilon} \sigma_{n3}(\boldsymbol{V}; \boldsymbol{y}, z)\, ds_y = p(z).$$

From this formula, the mechanical meaning of the function $p(z)$ is the density of linear tangential stresses exerted on the matrix by the inclusion.

At the ends of the bar, the stress–strain state in the elastic matrix is substantially three-dimensional and is not described by the constructed equations of plane boundary layer. Approaches to constructing the corresponding three-dimensional boundary layers are considered in [5, 10].

4. Joining of the Internal and External Asymptotic Representations. Making the change of variables (3.1) in representation (3.7) and using relations (3.5) and (1.1), we obtain

$$V_3(\varepsilon^{-1}\boldsymbol{y}; z) = -\frac{1}{2\pi\mu}\, p(z) \ln\Big(\frac{|\boldsymbol{y}|}{r_\varepsilon}\Big). \tag{4.1}$$

At the same time, from formulas (2.7), (2.8), (2.5), and (2.2) for a fixed value of $z \in (-l, l)$ and for $|\boldsymbol{y}| \to 0$ with accuracy up to terms of order $O(C|\boldsymbol{y}|)$, we have

$$v_3(p; \boldsymbol{y}, z) = v_3^0(0, z) - \frac{2}{M}\, p(z)$$

$$+ \frac{4(1-\nu)}{M}\Big\{p(z)\Big[-2\ln\Big(\frac{|\boldsymbol{y}|}{2l}\Big) + \ln\Big(1 - \frac{z^2}{l^2}\Big)\Big] + (\boldsymbol{J}p)(z)\Big\} + \dots\,. \tag{4.2}$$

We recall that the joining procedure consists of determining the relating between the functions $p(z)$ and $v_3^0(0, z)$ that leads to the asymptotic relation

$$v_3(p; \boldsymbol{y}, z) - V_3(\varepsilon^{-1}\boldsymbol{y}; z) = o(1), \qquad |\boldsymbol{y}|/l \sim \sqrt{\varepsilon}, \qquad \varepsilon \to 0. \tag{4.3}$$

The condition that the asymptotic external representation of the displacement field of the elastic space written in expansion (4.2) coincides with its internal representation (4.1) leads to the equation

$$p(z)\Big[2\ln\Big(\frac{2l}{r_\varepsilon}\Big) - \frac{1}{2(1-\nu)} + \ln\Big(1 - \frac{z^2}{l^2}\Big)\Big] + (\boldsymbol{J}p)(z) = -\frac{2\pi E}{1+\nu}\, v_3^0(0, z). \tag{4.4}$$

As is known, by virtue of the properties of the integral operator $\boldsymbol{J}$ [see formula (2.9)], the problem of seeking the solution of the resultant equation (4.4) for small values of the parameter ε is an ill-posed problem [10, 19]. However, to construct the asymptotic representation of the solution of the original problem, it is sufficient to find the approximate [accurate to within $O(\varepsilon]$ solution of Eq. (4.4).

Remark 4.1. From [10, 19], it follows that Eq. (4.4) remains valid in the case of a bar Q_ε of variable thickness with cross-sectional radius $r_\varepsilon(z)$. Thus, for a thin ellipsoid of revolution with the maximum cross-sectional radius $r_\varepsilon(0)$ [see (1.1)], $r_\varepsilon(z) = r_\varepsilon\sqrt{1-(z/l)^2}$ and Eq. (4.4) is simplified to

$$\Lambda_\varepsilon p(z) + (\boldsymbol{J}p)(z) = -2\pi\sigma z/(1+\nu), \tag{4.5}$$

where

$$\Lambda_\varepsilon = 2\ln(2l/r_\varepsilon) - 1/[2(1-\nu)]. \tag{4.6}$$

It is easy to verify that the solution of Eq. (4.5) is expressed by the formula

$$p(z) = -\frac{2\pi\sigma}{1+\nu}\,\frac{z}{\Lambda_\varepsilon - 2}. \tag{4.7}$$

Remark 4.2. Equation (4.4) contains the large (at $\varepsilon \to 0$) parameter Λ_ε; therefore, it admits an asymptotic solution in the form of the expansion [19]

$$p(z) = \Lambda_\varepsilon^{-1} q^0(z) + \Lambda_\varepsilon^{-2} q^1(z) + \ldots\ . \tag{4.8}$$

Substitution of expansion (4.8) into Eq. (4.4) yields

$$q^0(z) = -\frac{2\pi E}{1+\nu}\, v_3^0(0,z), \quad q^i(z) = -q^{i-1}(z)\ln\Big(1-\frac{z^2}{l^2}\Big) - (\boldsymbol{J}q^{i-1})(z) \quad (i=1,2,\ldots). \tag{4.9}$$

We note that the solution obtained in [4, 5] corresponds to the main term (4.9) of the logarithmic asymptotic representation (4.8).

5. Equation for the Density *p*. In [15], a modified procedure was proposed to join the external (2.2) and internal (3.2) asymptotic representations without using the asymptotic formulas (2.8) and (4.2), which contains the integral operator $\boldsymbol{J}$. Thus, according to relations (2.5) and (2.7), for $|\boldsymbol{y}| \to 0$, instead of the asymptotic expansion (4.2), we have

$$v_3(p;\boldsymbol{y},z) = v_3^0(0,z) + \frac{4(1-\nu)}{M}\, I_1^0(p;\boldsymbol{y},z) - \frac{2}{M}\, p(z) + O\Big(C\sum_{\pm}\frac{|\boldsymbol{y}|^2}{(l\pm z)^2}\Big). \tag{5.1}$$

Using the asymptotic formula (2.8), it is easy to show that the joining condition (4.3) is satisfied if, for $|\boldsymbol{y}| = \sqrt{\varepsilon}\, l$, we equate the expressions written in expansions (5.1) and (4.1). As a result, we have the following equation [compare with (4.4)]:

$$\Lambda p(z) + (J_\varepsilon p)(z) = -2\pi E v_3^0(0,z)/(1+\nu). \tag{5.2}$$

Here

$$(J_\varepsilon p)(z) = \int_{-l}^{l} \frac{p(s)\,ds}{\sqrt{(z-s)^2 + r_\varepsilon l}}, \qquad \Lambda = \ln\Big(\frac{l}{r_\varepsilon}\Big) - \frac{1}{2(1-\nu)}. \tag{5.3}$$

The theorem of unique solvability of Eq. (5.2) is proved in [15].

Remark 5.1. Equations (5.2) and (4.4) are also valid for the case of a cylindrical bar Q_ε with cross section ω_ε of arbitrary shape (see also [14]). In this case, the quantity r_ε is equal to the external conformal radius of the closed region $\overline{\omega}_\varepsilon$ (see, for example, [20]).

6. Asymptotic Model for Deformation of a Rigid Fiber in an Elastic Matrix. We assume that the bar Q_ε is made of an elastic material with elastic modulus E_j. The ratio E_j/E will be considered large. In the absence of volume loads, the strain of the bar is described by the equation

$$E_j S_\varepsilon \frac{d^2 w}{dz^2}(z) = p(z), \qquad z \in (-l, l) \tag{6.1}$$

with the boundary conditions (the action of the extended matrix on the bar through its ends is ignored)

$$E_j S_\varepsilon \frac{dw}{dz}(\pm l) = 0. \tag{6.2}$$

Here S_ε is the cross-sectional area ω_ε and $w(z)$ is the displacement of the bar cross section with the coordinate z.

The equation relating the function $w(z)$ to the response $p(z)$ is derived using the method proposed in [13, 14]. The assumption of complete compatibility of the fiber with the matrix corresponds to the boundary condition $V_3(\boldsymbol{\eta}; z) = w(z)$ for $\boldsymbol{\eta} \in \partial\omega_1$ [compare with (3.4)]. This leads to the new expression for the nontrivial component of the plane boundary layer [compare with (3.7)]:

$$V_3(\boldsymbol{\eta}; z) = \mu^{-1} p(z) G_\infty(\boldsymbol{\eta}) + w(z). \tag{6.3}$$

The joining of the external asymptotic representation (2.2) with the internal representation (3.2), (6.3) results in the equation

$$p(z)[\Lambda_\varepsilon + \ln(1 - z^2/l^2)] + (\boldsymbol{J}p)(z) = 2\pi E[w(z) - v_3^0(0, z)]/(1 + \nu)$$

[the parameter Λ_ε and the integral operator $\boldsymbol{J}$ are defined in (4.6) and (2.9)].

Using the modified joining procedure [15], we obtain the equation

$$\Lambda p(z) + (J_\varepsilon p)(z) = 2\pi E[w(z) - v_3^0(0, z)]/(1 + \nu) \tag{6.4}$$

[the parameter Λ and the integral operator J_ε are defined in (5.3)].

The results of numerical calculations using the asymptotic model (6.1), (6.2), (6.4) are compared with the results of calculations [1] using the finite element method. The following parameters were calculated: the tangential stress on the bar surface $\tau(z)$ and the stress averaged over the bar cross section $\sigma_j(z)$:

$$\tau(z) = -\frac{1}{2\pi r_\varepsilon} p(z), \qquad \sigma_j(z) = \frac{1}{\pi r_\varepsilon^2} \int_{-l}^{z} p(s)\, ds.$$

The calculations were performed for $\varepsilon = 0.01$ and $E/E_j = 10^{-4}$; 10^{-5}. The difference between the obtained values of $\tau(z)$ and $\sigma_j(z)$ and the calculation results [1] is 5 and 14% respectively.

Possible methods of generalizing the mathematical model constructed for deformation of an elastic matrix with a rigid bar are indicated in Remarks 4.1 and 5.1. The formulas obtained can be used to calculate the optimum length of the reinforcing fiber of constant section (see also [1]).

7. Shape Optimization of the Reinforcing Bar Based on the Uniform Strength Condition. We use the main term of the logarithmic asymptotic (4.8) and set

$$p(z) = -\frac{1}{\Lambda(z)} \frac{2\pi\sigma}{1 + \nu} z. \tag{7.1}$$

Considering the bar radius $r(z)$ variable, according to formula (4.6), we have

$$\Lambda(z) = 2\ln\Big(\frac{2l}{r(z)}\Big) - \frac{1}{2(1 - \nu)}. \tag{7.2}$$

It is assumed that $\max r(z) \ll l$ for $|z| \le l$.

We find the function $r(z)$ from the uniform strength condition for the reinforcing bar. Denoting the admissible stress by σ_0, we obtain

$$\frac{1}{\pi r^2(z)} \int_{-l}^{z} p(s)\, ds = \sigma_0, \tag{7.3}$$

whence follows the relation

$$p(z) = 2\pi\sigma_0 r(z) r'(z). \tag{7.4}$$

Substituting expression (7.4) into Eq. (7.1) and using (7.2), we obtain the equation

$$2\pi\sigma_0\Big(2\ln\Big(\frac{2l}{r}\Big) - \frac{1}{2(1 - \nu)}\Big) r \frac{dr}{dz} = -\frac{2\pi\sigma}{1 + \nu} z.$$

Dividing the variables and integrating, we have

$$r^2(z)\Big(2\ln\Big(\frac{2l}{r(z)}\Big) - \frac{1}{2(1-\nu)} + 1\Big) = \frac{\sigma}{(1+\nu)\sigma_0}\,(l^2 - z^2). \tag{7.5}$$

It should be noted that the integration constant in integral (7.5) is chosen so as to satisfy the boundary condition $r(\pm l) = 0$, which follows from condition (7.3).

The main term of the logarithmic asymptotic representation of the solution of Eq. (7.5) defines the optimum shape —the ellipsoid of revolution:

$$r(z) = r_0\sqrt{1 - z^2/l^2}. \tag{7.6}$$

We show that relation (7.6) is the exact solution of the optimization problem considered if Eq. (4.4) is used instead of the simplified equation (7.1). Indeed, substitution of expression (7.6) into Eq. (4.4) leads to the following relation [see also formulas (4.5) and (4.7)]:

$$p(z) = -\frac{2\pi\sigma}{1+\nu}\frac{z}{\Lambda_0 - 2}. \tag{7.7}$$

Here

$$\Lambda_0 = 2\ln\Big(\frac{2l}{r_0}\Big) - \frac{1}{2(1-\nu)}.$$

Substitution of expressions (7.6) and (7.7) into relation (7.3) yields

$$\frac{r_0^2}{l^2}\Big(2\ln\Big(\frac{2l}{r_0}\Big) - \frac{1}{2(1-\nu)} - 2\Big) = \frac{\sigma}{(1+\nu)\sigma_0}. \tag{7.8}$$

The transcendental equation (7.8) allows us to determine the value of the parameter r_0 in formula (7.6).

Finally, we show that the ellipsoidal shape of the reinforcing bar (7.6) is also optimal if we take into account the fiber deformation in the asymptotic model constructed, which is defined by the relations

$$p(z)\Big[2\ln\Big(\frac{2l}{r(z)}\Big) - \frac{1}{2(1-\nu)} + \ln\Big(1 - \frac{z^2}{l^2}\Big)\Big] + (\boldsymbol{J}p)(z) = \frac{2\pi E}{1+\nu}\Big(w(z) - \frac{\sigma z}{E}\Big),$$
$$\frac{d}{dz}\Big(E_j\pi r^2(z)\,\frac{dw}{dz}(z)\Big) = p(z), \qquad z \in (-l, l), \tag{7.9}$$
$$E_j\pi r^2(\pm l)\,\frac{dw}{dz}(\pm l) = 0.$$

In this case, the uniform strength condition of the bar becomes

$$\frac{1}{\pi r^2(z)}\Big(E_j\pi r^2(z)\,\frac{dw}{dz}(z)\Big) = \sigma_0. \tag{7.10}$$

We set

$$p(z) = -p_0 z/l, \qquad w(z) = w_0 z/l. \tag{7.11}$$

Then, in view of (7.6) and (7.11), relations (7.9) are satisfied exactly if the following equalities hold:

$$\frac{p_0}{l}\Big(2\ln\Big(\frac{2l}{r_0}\Big) - \frac{1}{2(1-\nu)} - 2\Big) = \frac{2\pi E}{1+\nu}\Big(\frac{\sigma}{E} - \frac{w_0}{l}\Big), \qquad 2\pi E_j w_0\,\frac{r_0^2}{l^2} = p_0. \tag{7.12}$$

Substitution of expressions (7.6) and (7.11) into Eq. (7.10) leads to the equation

$$E_j w_0/l = \sigma_0. \tag{7.13}$$

Eliminating the parameters p_0 and w_0 from system (7.12), (7.13), we have the following equation for the parameter r_0:

$$\frac{r_0^2}{l^2}\Big(2\ln\Big(\frac{2l}{r_0}\Big) - \frac{1}{2(1-\nu)} - 2\Big) = \frac{\sigma}{(1+\nu)\sigma_0} - \frac{E}{(1+\nu)E_j}. \tag{7.14}$$

Equation (7.14) implies that accounting for the deformation of the reinforcing bar leads to a decrease in the parameter r_0.

Conclusions. The shape optimization problem was solved by constructing an asymptotic model for deformation of a rigid inclusion of variable section in an elastic medium based on the uniform strength criterion. The optimum shape of the reinforcing rigid fiber was found to be an ellipsoid. Fibers of finite length (frequently modeled by prolate ellipsoids) are used as reinforcing elements in modern composite materials (see, for example, a review [21]). As is known, the greatest reinforcing effect is produced by inclusions that have much higher rigidity (than that of the matrix).

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